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Geometric crystals on Schubert varieties

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Abstract

We define geometric crystals and unipotent crystals for arbitrary Kac–Moody groups and describe geometric and unipotent crystal structures on the Schubert varieties. We give some examples in affine $\widehat{\mathfrak{sl}}_2$-case.

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1. Introduction

The theory of crystal base introduced by Kashiwara succeeds in being applied to many areas in mathematics and mathematical physics to clarify their combinatorial behavior. One of the reasons why it can be well-applied is that it allows not only “real crystals” but also “virtual crystals”, e.g., B_i (see Section 6), B_∞ (see Section 7), t_λ (see [7]),etc, where “virtual crystals” mean certain crystals not having the corresponding $U_q(\mathfrak{g})$-modules, which are purely combinatorial objects. Some of them are obtained as ‘limit’ of real crystals and they have good combinatorial properties, e.g., the crystals B_i and B_∞ are regarded as some

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limits of real crystals. ‘Real one’ and ‘virtual’ one are deeply related to each other. Indeed, the “real crystal” $B(\lambda)$ (resp. $B(\infty)$) is described as a subcrystal in infinitely many tensor products of “virtual crystals” B_i’s (see [17–19]), where $B(\lambda)$ (resp. $B(\infty)$) is a crystal of the irreducible integrable highest weight module $V(\lambda)$ (resp. a crystal of the subalgebra $U_q^-(\mathfrak{g})$). In this sense, the theory of crystals would cover wider area than usual representation theory of the quantum algebra $U_q(\mathfrak{g})$ does. Roughly, we can say that real crystal bases are obtained by taking the limit $q \to 0$ from some bases of $U_q(\mathfrak{g})$-modules, which is called “crystallization”. But, “virtual” ones are not gotten by such crystallization procedure from $U_q(\mathfrak{g})$-modules.

Berenstein and Kazhdan clarify [1] that such “virtual crystals” also have some “real” backgrounds as the “tropicalization/ultra-discretization” of “geometric crystals” for semi-simple(reductive) groups.

$$\begin{tikzcd}
 & \{\text{Global Bases of } U_q(\mathfrak{g})\text{-modules}\} \arrow[dl, dashed, leftrightarrow] \arrow[dr, shift left, "\text{crystallization}"] & \\
\{\text{Geometric Crystals}\} \arrow[rr, shift left, "\text{ultra-discretization}"] & & \{\text{Crystals}\} \arrow[ul, shift left, "\text{melting}"] \arrow[ll, shift left, "\text{tropicalization}"]
\end{tikzcd}$$

Recently, by the ultra-discretization/tropicalization method, the relations between soliton cellular automaton and crystals are revealed (see e.g., [4,5]). In the meanwhile, it is well-known that flag varieties G/B (reps. G/P) plays a significant role in the soliton theory, where G is an affine Kac–Moody group and B (resp. P) is its Borel (resp. parabolic) subgroup. We would like to find the connection of affine flag varieties and geometric crystals. For the purpose, we shall extend the theory of geometric/unipotent crystals [1] to Kac–Moody setting. And then we shall define geometric/unipotent crystals on Schubert cells/varieties associated with Kac–Moody groups. We consider some ‘positive structures’ on them (see Section 5), and we show that some ultra-discretizations of the geometric crystals on Schubert varieties are isomorphic to tensor products of some Kashiwara’s crystals. These results are simple generalizations of the results in [1] for reductive setting to the one for the Kac–Moody setting. Thus, in order to show the validity of the extension to Kac–Moody settings, we shall present an interesting example for affine $\widehat{\mathfrak{sl}}_2$-case by showing that some geometric crystals on affine Schubert cells/varieties are related to “perfect crystals”. They are affine crystals associated with quantum affine algebras and play an important role in studying vertex type solvable lattice models [9,10]. There exists some “limit” of perfect crystals [8] denoted by B_∞ and we shall see that an ultra-discretization of certain geometric crystal on affine Schubert varieties coincides with this B_∞ for $\widehat{\mathfrak{sl}}_2$-case. As for higher rank affine cases, we will discuss in forthcoming papers.

The organization of the article is as follows; in Section 2 we review briefly the theory of Kac–Moody groups, ind-varieties and ind-groups. In Section 3, we define the notion of unipotent crystals in Kac–Moody setting and their product structures. We also define the notion of geometric crystals and give a recipe for obtaining canonically geometric crystals from unipotent crystals following [1]. In Section 4, on finite Schubert cells/varieties we induce the structure of unipotent/geometric crystals. In Section 5, we recall the notion of positive structure on geometric crystals and define ultra-discretization/tropicalization operations. We also consider certain positive structure of geometric crystals on Schubert cells and show that its ultra-discretization is isomorphic to (Langlands dual of) Kashiwara's crystal $B_{i_1} \otimes \cdots \otimes B_{i_l}$. In Section 6, we apply the result in Section 5 to give a new proof of braid-type isomorphisms [19]. In the last section, we shall see how to relate certain geometric crystal on an affine Schubert cell and the limit of perfect crystal B_∞ for $\widehat{\mathfrak{sl}}_2$-case.

2. Kac–Moody groups and Ind-varieties

In this section, we review Kac–Moody groups following [13,15,22].

2.1. Kac–Moody algebras and Kac–Moody groups

Fix a symmetrizable generalized Cartan matrix $A = (a_{ij})_{i,j\in I}$, where I be a finite index set. Let $(\mathfrak{t}, \{\alpha_i\}_{i\in I}, \{h_i\}_{i\in I})$ be the associated root data, where $\mathfrak{t}$ be the vector space over $\mathbb{C}$ with dimension $|I|+$ corank(A), and $\{\alpha_i\}_{i\in I} \subset \mathfrak{t}^*$ and $\{h_i\}_{i\in I} \subset \mathfrak{t}$ are linearly independent indexed sets satisfying $\alpha_j(h_i) = a_{ij}$.

The Kac–Moody Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ associated with A is the Lie algebra over $\mathbb{C}$ generated by $\mathfrak{t}$, the Chevalley generators e_i and f_i $(i \in I)$ with the usual defining relations [13,15]. There is the root space decomposition $\mathfrak{g} = \bigoplus_{\alpha\in\mathfrak{t}^*} \mathfrak{g}_\alpha$. Denote the set of roots by $\Delta := \{\alpha \in \mathfrak{t}^* | \alpha \neq 0, \mathfrak{g}_\alpha \neq (0)\}$. Set $Q = \sum_i \mathbb{Z}\alpha_i$, $Q_+ = \sum_i \mathbb{Z}_{\geq 0}\alpha_i$ and $\Delta_+ := \Delta \cap Q_+$. An element of Δ_+ is called a positive root.

Define simple reflections $s_i \in \mathrm{Aut}(\mathfrak{t})$ $(i \in I)$ by $s_i(h) := h - \alpha_i(h)h_i$, which generate the Weyl group W. We also define the action of W on $\mathfrak{t}^*$ by $s_i(\lambda) := \lambda - \alpha(h_i)\alpha_i$. Set $\Delta^{\mathrm{re}} := \{w(\alpha_i) | w \in W, i \in I\}$, whose element is called a real root.

Let $\mathfrak{g}'$ be the derived Lie algebra of $\mathfrak{g}$ and G^* be the free group generated by the free product of the additive groups $\mathfrak{g}_\alpha$ $(\alpha \in \Delta^{\mathrm{re}})$, with the canonical inclusion $i_\alpha : \mathfrak{g}_\alpha \hookrightarrow G^*$. For any integrable $\mathfrak{g}'$-module (V, π), a homomorphism $\pi_V^* : G^* \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V)$ is defined by $\pi_V^*(i_\alpha(e)) = \exp \pi(e)$. Set $N^* := \cap_{V:\text{integrable}} \mathrm{Ker}(\pi_V^*)$ and $G := G^*/N^*$, which is called a Kac–Moody group associated with the Kac–Moody Lie algebra $\mathfrak{g}'$. Let $\rho : G^* \to G$ be the canonical homomorphism. For $e \in \mathfrak{g}_\alpha$ $(\alpha \in \Delta^{\mathrm{re}})$, define $\exp e := \rho(i_\alpha(e))$ and $U_\alpha := \exp \mathfrak{g}_\alpha$, which is a one-parameter subgroup of G. The group G is generated by $U_\alpha(\alpha \in \Delta^{\mathrm{re}})$. Let $U^\pm$ be the subgroups generated by $U_{\pm\alpha}$ $(\alpha \in \Delta_+^{\mathrm{re}} = \Delta^{\mathrm{re}} \cap Q_+)$, i.e., $U^\pm := \langle U_{\pm\alpha} | \alpha \in \Delta_+^{\mathrm{re}} \rangle$.

For any $i \in I$, there exists a unique homomorphism; $\phi_i : SL_2(\mathbb{C}) \to G$ such that

$$\phi_i\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right) = \exp te_i, \qquad \phi_i\left(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\right) = \exp\ tf_i (t \in \mathbb{C}).$$

Set $G_i := \phi_i(SL_2(\mathbb{C}))$,

$x_i(t) := \exp\ te_i$, $y_i(t) := \exp\ tf_i$, $T_i := \phi_i(\{\mathrm{diag}(t, t^{-1}) | t \in \mathbb{C}\})$ and $N_i := N_{G_i}(T_i)$. Let T (resp. N) be the subgroup of G generated by T_i (resp. N_i), which is called a *maximal torus* in G and $B^\pm = U^\pm T$ be the Borel subgroup of G. We have the isomorphism $\phi : W \xrightarrow{\sim} N/T$ defined by $\phi(s_i) = N_i T/T$. An element $\bar{s}_i := x_i(-1)y_i(1)x_i(-1)$ is in $N_G(T)$, which is a representative of $s_i \in W = N_G(T)/T$. Define $R(w)$ for $w \in W$ by

$$R(w) := \{(i_1, i_2, \dots, i_l) \in I^l | w = s_{i_1} s_{i_2} \cdots s_{i_l}\},$$

where l is the length of w. We associate to each $w \in W$ its standard representative $\overline{w} \in N_G(T)$ by

$$\overline{w} = \overline{s}_{i_1} \overline{s}_{i_2} \cdots \overline{s}_{i_l},$$

for any $(i_1, i_2, \dots, i_l) \in R(w)$.

2.2. Ind-variety and Ind-group

Let us recall the notion of ind-varieties and ind-groups (see [12]).

Definition 2.1. Let k be an algebraically closed field.

(i) A set X is an *ind-variety* over k if there exists a filtration $X_0 \subset X_1 \subset X_2 \subset \cdots$ such that
  (a) $\bigcup_{n \geq 0} X_n = X$.
  (b) Each X_n is a finite-dimensional variety over k such that the inclusion $X_n \hookrightarrow X_{n+1}$ is a closed embedding.
  The ring of regular functions $k[X]$ is defined by
  $k[X] := \varprojlim_n k[X_n]$.

(ii) A *Zariski topology* on an ind-variety X is defined as follows; a set $U \subset X$ is open if and only if $U \cap X_n$ is open in X_n for any $n \geq 0$.

(iii) Let X and Y be two ind-varieties with filtrations $\{X_n\}$ and $\{Y_n\}$ respectively. A map $f : X \to Y$ is a *morphism* if for any $n \geq 0$, there exists m such that $f(X_n) \subset Y_m$ and $f_{X_n} : X_n \to Y_m$ is a morphism. A morphism $f : X \to Y$ is said to be an *isomorphism* if f is bijective and $f^{-1} : Y \to X$ is also a morphism.

(iv) Let X and Y be two ind-varieties. A *rational morphism* $f : X \to Y$ is an equivalence class of morphisms $f_U : U \to Y$ where U is an open dense subset of X, and two morphisms $f_U : U \to Y$ and $f_V : V \to Y$ are equivalent if they coincide on $U \cap V$.

Lemma 2.2.

(i) *A finite dimensional variety over k holds canonically an ind-variety structure.*

(ii) *If X and Y are ind-varieties, then $X \times Y$ is canonically an ind-variety by taking the filtration*

$$(X \times Y)_n := X_n \times Y_n.$$

Definition 2.3. An ind-variety H is called an *ind* (*algebraic*)-*group* if the underlying set H is a group and the maps

$$\begin{array}{ll} H \times H \longrightarrow H & H \longrightarrow H \\ (x, y) \mapsto xy & x \mapsto x^{-1} \end{array}$$

are morphisms of ind-varieties.

Proposition 2.4. (Kumar [12])

(i) *Let G be a Kac–Moody group and $U^{\pm}$, $B^{\pm}$ be its subgroups as above. Then G is an ind-group and $U^{\pm}$, $B^{\pm}$ are its closed ind-subgroups.*
(ii) *The multiplication maps*

$$\begin{array}{ll} T \times U \longrightarrow B & U^{-} \times T \longrightarrow B^{-} \\ (t, u) \mapsto tu & (v, t) \mapsto vt \end{array}$$

are isomorphisms of ind-varieties.

3. Geometric crystals and unipotent crystals

In this section, we define geometric crystals and unipotent crystals associated with Kac–Moody groups, which is just a generalization of [1] to a Kac–Moody setting.

3.1. Geometric crystals

Let $(a_{ij})_{i,j\in I}$ be a symmetrizable generalized Cartan matrix and G be the associated Kac–Moody group with the maximal torus T. An element in $\mathrm{Hom}(T, \mathbb{C}^{\times})$ (resp. $\mathrm{Hom}(\mathbb{C}^{\times}, T)$) is called a *character* (resp. *co-character*) of T. We define a *simple co-root* $\alpha_i^{\vee} \in \mathrm{Hom}(\mathbb{C}^{\times}, T)$ $(i \in I)$ by $\alpha_i^{\vee}(t) := T_i$. We have a pairing $\langle \alpha_i^{\vee}, \alpha_j \rangle = a_{ij}$.

Definition 3.1.

(i) Let X be an ind-variety over $\mathbb{C}$, $\gamma : X \to T$ be a rational morphism and a family of rational $\mathbb{C}$-actions $e_i : \mathbb{C}^{\times} \times X \to X$ $(i \in I)$;

$$\begin{aligned} e_i : \mathbb{C}^{\times} \times X &\longrightarrow X \\ (c, x) &\mapsto e_i^c(x). \end{aligned}$$

The triplet $\chi = (X, \gamma, \{e_i\}_{i\in I})$ is a *geometric pre-crystal* if it satisfies $\{1\} \times X \subset \mathrm{dom}(e_i)$, $e^1(x) = x$ and

$$\gamma(e_i^c(x)) = \alpha_i^{\vee}(c)\gamma(x). \tag{3.1}$$

(ii) Let $(X, \gamma_X, \{e_i^X\}_{i\in I})$ and $(Y, \gamma_Y, \{e_i^Y\}_{i\in I})$ be geometric pre-crystals. A rational morphism $f : X \to Y$ is a *morphism of geometric pre-crystals* if f satisfies that

$$f \circ e_i^X = e_i^Y \circ f, \quad \gamma_X = \gamma_Y \circ f.$$

In particular, if a morphism f is a birational isomorphism of ind-varieties, it is called an *isomorphism of geometric pre-crystals*.

Let $\chi = (X, \gamma, \{e_i\}_{i\in I})$ be a geometric pre-crystal. For a word $\mathbf{i} = (i_1, i_2, \ldots, i_l) \in R(w)$ $(w \in W)$, set $\alpha^{(l)} := \alpha_{i_l}, \alpha^{(l-1)} := s_{i_l}(\alpha_{i_{l-1}}), \ldots, \alpha^{(1)} := s_{i_l}\cdots s_{i_2}(\alpha_{i_1})$. Now for a word $\mathbf{i} = (i_1, i_2, \ldots, i_l) \in R(w)$ we define a rational morphism $e_{\mathbf{i}} : T \times X \to X$ by

$$(t, x) \mapsto e_{\mathbf{i}}^t(x) := e_{i_1}^{\alpha^{(1)}(t)} e_{i_2}^{\alpha^{(2)}(t)} \cdots e_{i_l}^{\alpha^{(l)}(t)}(x).$$

Definition 3.2.

(i) A geometric pre-crystal χ is called a *geometric crystal* if for any $w \in W$, and any $\mathbf{i}$, $\mathbf{i}' \in R(w)$ we have

$$e_{\mathbf{i}} = e_{\mathbf{i}'}. \tag{3.2}$$

(ii) Let $(X, \gamma_X, \{e_i^X\}_{i\in I})$ and $(Y, \gamma_Y, \{e_i^Y\}_{i\in I})$ be geometric crystals. A rational morphism $f : X \to Y$ is called a *morphism* (resp. an *isomorphism*) *of geometric crystals* if it is a morphism (resp. an isomorphism) of geometric pre-crystals.

The following lemma is a direct result from [1, Lemma 2.1] and the fact that the Weyl group of any Kac–Moody Lie algebra is a Coxeter group [6, Proposition 3.13].

Lemma 3.3. *The relations* (3.2) *are equivalent to the following relations*:

$$\begin{aligned}
&e_i^{c_1} e_j^{c_2} = e_j^{c_2} e_i^{c_1} && \text{if } \langle\alpha_i^\vee, \alpha_j\rangle = 0,\\
&e_i^{c_1} e_j^{c_1c_2} e_i^{c_2} = e_j^{c_2} e_i^{c_1c_2} e_j^{c_1} && \text{if } \langle\alpha_i^\vee, \alpha_j\rangle = \langle\alpha_j^\vee, \alpha_i\rangle = -1,\\
&e_i^{c_1} e_j^{c_1^2c_2} e_i^{c_1c_2} e_j^{c_2} = e_j^{c_2} e_i^{c_1c_2} e_j^{c_1^2c_2} e_i^{c_1} && \text{if } \langle\alpha_i^\vee, \alpha_j\rangle = -2, \langle\alpha_j^\vee, \alpha_i\rangle = -1,\\
&e_i^{c_1} e_j^{c_1^2c_2} e_i^{c_1^3c_2} e_j^{c_1^3c_2^2} e_i^{c_1c_2} e_j^{c_2} = e_j^{c_2} e_i^{c_1c_2} e_j^{c_1^3c_2^2} e_i^{c_1^3c_2} e_j^{c_1^2c_2} e_i^{c_1} && \text{if } \langle\alpha_i^\vee, \alpha_j\rangle = -3, \langle\alpha_j^\vee, \alpha_i\rangle = -1.
\end{aligned}$$

Remark. If $\langle\alpha_i^\vee, \alpha_j\rangle\langle\alpha_j^\vee, \alpha_i\rangle \geq 4$, there is no relation between e_i and e_j.

3.2. Unipotent crystals

In the sequel, we denote the unipotent subgroup U^+ by U. We define unipotent crystals (see [1]) associated to Kac–Moody groups.

The definitions below are as in [1].

Definition 3.4. Let X be an ind-variety over $\mathbb{C}$ and $\alpha : U \times X \to X$ be a rational U-action such that α is defined on $\{e\} \times X$. Then, the pair $\mathbf{X} = (X, \alpha)$ is called a *U-variety*. For

U-varieties $\mathbf{X} = (X, \alpha_X)$ and $\mathbf{Y} = (Y, \alpha_Y)$, a rational morphism $f : X \to Y$ is called a *U-morphism* if it commutes with the action of U.

Now, we define the U-variety structure on $B^- = U^- T$. By Proposition 2.4, B^- is an ind-subgroup of G and hence an ind-variety over $\mathbb{C}$. The multiplication map in G induces the open embedding; $B^- \times U \hookrightarrow G$, which is a birational isomorphism. Let us denote the inverse birational isomorphism by g;

$$g : G \longrightarrow B^- \times U.$$

Then we define the rational morphisms $\pi^- : G \to B^-$ and $\pi : G \to U$ by $\pi^- := \mathrm{proj}_{B^-} \circ g$ and $\pi := \mathrm{proj}_U \circ g$. Now we define the rational U-action α_{B^-} on B^- by

$$\alpha_{B^-} := \pi^- \circ m : U \times B^- \longrightarrow B^-,$$

where m is the multiplication map in G. Then we obtain U-variety $\mathbf{B}^- = (B^-, \alpha_{B^-})$.

Definition 3.5.

(i) Let $\mathbf{X} = (X, \alpha)$ be a U-variety and $f : X \to \mathbf{B}^-$ be a U-morphism. The pair $(\mathbf{X}, f)$ is called a *unipotent G-crystal* or, for short, *unipotent crystal*.

(ii) Let $(\mathbf{X}, f_X)$ and $(\mathbf{Y}, f_Y)$ be unipotent crystals. A U-morphism $g : X \to Y$ is called a *morphism of unipotent crystals* if $f_X = f_Y \circ g$. In particular, if g is a birational isomorphism of ind-varieties, it is called an *isomorphism of unipotent crystals*.

We define a product of unipotent crystals following [1]. For unipotent crystals $(\mathbf{X}, f_X)$, $(\mathbf{Y}, f_Y)$, define a morphism $\alpha_{X\times Y} : U \times X \times Y \to X \times Y$ by

$$\alpha_{X\times Y}(u, x, y) := (\alpha_X(u, x), \alpha_Y(\pi(u \cdot f_X(x)), y)). \tag{3.3}$$

If there is no confusion, we use abbreviated notation $u(x, y)$ for $\alpha_{X\times Y}(u, x, y)$.

Theorem 3.6 (Berenstein and Kazhdan [1]).

(i) *The morphism $\alpha_{X\times Y}$ defined above is a rational U-morphism on $X \times Y$.*

(ii) *Let $\mathbf{m} : B^- \times B^- \to B^-$ be a multiplication morphism and $f = f_{X\times Y} : X \times Y \to B^-$ be the rational morphism defined by*

$$f_{X\times Y} := \mathbf{m} \circ (f_X \times f_Y).$$

Then $f_{X\times Y}$ is a U-morphism and $(\mathbf{X} \times \mathbf{Y}, f_{X\times Y})$ is a unipotent crystal, which we call a product of unipotent crystals $(\mathbf{X}, f_X)$ and $(\mathbf{Y}, f_Y)$.

(iii) *Product of unipotent crystals is associative.*

3.3. From unipotent crystals to geometric crystals

For $i \in I$, set $U_i^{\pm} := U^{\pm} \cap \overline{s}_i U^{\mp} \overline{s}_i^{-1}$ and $U_{\pm}^i := U^{\pm} \cap \overline{s}_i U^{\pm} \overline{s}_i^{-1}$. Indeed, $U_i^{\pm} = U_{\pm\alpha_i}$. Set

$$Y_{\pm\alpha_i} := \langle x_{\pm\alpha_i}(t) U_\alpha x_{\pm\alpha_i}(-t) | t \in \mathbb{C}, \quad \alpha \in \Delta_{\pm}^{\mathrm{re}} \setminus \{\pm\alpha_i\}\rangle.$$

Lemma 3.7 (Kumar et al. [12,14]). *For a simple root α_i ($i \in I$), we have*:

(i) $Y_{\pm\alpha_i} = U_{\pm}^i$.
(ii) $U^{\pm} = U_i^{\pm} \cdot Y_{\pm\alpha_i}$ (*semi-direct product*).
(iii) $\overline{s}_i Y_{\pm\alpha_i} \overline{s}_i^{-1} = Y_{\pm\alpha_i}$.

By this lemma, we have the unique decomposition;

$$U^- = U_i^- \cdot Y_{\pm\alpha_i} = U_{-\alpha_i} \cdot U_-^i.$$

By using this decomposition, we get the canonical projection $\xi_i : U^- \to U_{-\alpha_i}$. Now, we define the function on U^- by

$$\chi_i := y_i^{-1} \circ \xi_i : U^- \longrightarrow U_{-\alpha_i} \longrightarrow \mathbb{C},$$

and extend this to the function on B^- by $\chi_i(u \cdot t) := \chi_i(u)$ for $u \in U^-$ and $t \in T$. For a unipotent G-crystal $(\mathbf{X}, \mathbf{f_X})$, we define a function $\varphi_i := \varphi_i^X : X \to \mathbb{C}$ by

$$\varphi_i := \chi_i \circ \mathbf{f_X},$$

and a rational morphism $\gamma_X : X \to T$ by

$$\gamma_X := \mathrm{proj}_T \circ \mathbf{f_X} : X \to B^- \to T, \tag{3.4}$$

where proj_T is the canonical projection. Suppose that the function φ_i is not identically zero on X. We define a morphism $e_i : \mathbb{C}^\times \times X \to X$ by

$$e_i^c(x) := x_i\left(\frac{c-1}{\varphi_i(x)}\right)(x). \tag{3.5}$$

Theorem 3.8 ([1]). *For a unipotent G-crystal $(\mathbf{X}, \mathbf{f_X})$, suppose that the function φ_i is not identically zero for any $i \in I$. Then the rational morphisms $\gamma_X : X \to T$ and $e_i : \mathbb{C}^\times \times X \to X$ as above define a geometric G-crystal $(X, \gamma_X, \{e_i\}_{i \in I})$, which is called the induced geometric G-crystals by unipotent G-crystal $(\mathbf{X}, f_X)$.*

Note that in [1], the cases $\varphi_i \equiv 0$ for some $i \in I$ are treated by considering Levi subgroups of G. But here we do not treat such things.

The following product structure on geometric crystals are most important results in the sense of comparison with the tensor product theorem in Kashiwara's crystal theory.

Proposition 3.9. *For unipotent G-crystals* $(\mathbf{X}, f_X)$ *and* $(\mathbf{Y}, f_Y)$, *set the product* $(\mathbf{Z}, f_Z) := (\mathbf{X}, f_X) \times (\mathbf{Y}, f_Y)$, *where* $Z = X \times Y$. *Let* $(Z, \gamma_Z, \{e_i\})$ *be the induced geometric G-crystal from* $(\mathbf{Z}, f_Z)$. *Then we obtain*:

(i) $\gamma_Z = \mathbf{m} \circ (\gamma_X \times \gamma_Y)$.
(ii) *For each* $i \in I$, $(x, y) \in Z$:

$$\varphi_i^Z(x, y) = \varphi_i^X(x) + \frac{\varphi_i^Y(y)}{\alpha_i(\gamma_X(x))}. \tag{3.6}$$

(iii) *For any* $i \in I$, *the action* $e_i : \mathbb{C}^\times \times Z \to Z$ *is given by*: $e_i^c(x, y) = (e_i^{c_1}(x), e_i^{c_2}(y))$, *where*

$$c_1 = \frac{c\alpha_i(\gamma_X(x))\varphi_i^X(x) + \varphi_i^Y(y)}{\alpha_i(\gamma_X(x))\varphi_i^X(x) + \varphi_i^Y(y)}, \qquad c_2 = \frac{\alpha_i(\gamma_X(x))\varphi_i^X(x) + \varphi_i^Y(y)}{\alpha_i(\gamma_X(x))\varphi_i^X(x) + c^{-1}\varphi_i^Y(y)}. \tag{3.7}$$

Here note that $c_1 c_2 = c$. The formula c_1 and c_2 in [1] seem to be different from ours. Thus, we give the proof of (iii). Others are obtained by the same way as in [1].

Proof. By using the result (ii), we have

$$\varphi_i^Z(x, y) = \varphi_i^X(x) + \frac{\varphi_i^Y(y)}{\alpha_i(\gamma_X(x))}.$$

Here we set $A := (c-1)/\varphi_i^Z(x, y)$ for $(x, y) \in Z$. Since by (3.3) we have

$$e_i^c(x, y) = x_i(A)(x, y) = (x_i(A)(x), \pi(x_i(A) \cdot f_X(x))(y)),$$

we get $(c_1 - 1)/\varphi_i^X(x) = A$, and then we obtain c_1 in (3.7).

Let us see c_2. Writing $f_X(x) = u \cdot t$ ($u \in U^-$, $t \in T$), by Lemma 3.1 (3.2) in [1], we get

$$\pi(x_i(A) \cdot f_X(x)) = x_i((A^{-1} + \chi_i(u)^{-1})^{-1}\alpha_i(t^{-1}))$$

Since $\chi_i(u) = \varphi_i(x)$ and $\alpha_i(t) = \alpha_i(\gamma_X(x))$, we obtain

$$\pi(x_i(A) \cdot f_X(x)) = x_i\left(\frac{A}{(1 + A\varphi_i(x))\alpha_i(\gamma_X(x))}\right).$$

Now, set $B = A/(1 + A\varphi_i^X(x))\alpha_i(\gamma_X(x))$. Substituting $A = (c-1)/\varphi_i^Z(x, y)$ and $(c_2 - 1)/\varphi_i^Y(y) = B$, we obtain the formula c_2 in (3.7). □

4. Crystal structure on Schubert varieties

4.1. Highest weight modules and Schubert varieties

As in Section 2, let G be a Kac–Moody group, $B^{\pm} = U^{\pm}T$ (resp. $U^{\pm}$) be the Borel (resp. unipotent) subgroups in G and W be the associated Weyl group. Here, we have the following Bruhat decomposition and Birkhoff decomposition.

Proposition 4.1 ([12,15,22]). *We have*

$$G = \bigcup_{w\in W} B^{+}\overline{w}B^{+} = \bigcup_{w\in W} U^{+}\overline{w}B^{+} \quad (\textit{Bruhat decomposition}), \tag{4.1}$$

$$G = \bigcup_{w\in W} B^{-}\overline{w}B^{+} = \bigcup_{w\in W} U^{-}\overline{w}B^{+} \quad (\textit{Birkhoff decomposition}). \tag{4.2}$$

Let $J \subset I$ be a subset of the index set I and $W_J := \langle s_i | i \in J\rangle$ be the subgroup of W associated with J. Set $P_J := B^{+}W_J B^{+}$ and call it a (standard) *parabolic subgroup* of G associated with $J \subset I$. The following lemma is well-known.

Lemma 4.2. *Any coset in W/W_J contains a unique element w^* of minimal length, and for any $w' \in W_J$, we have $l(w^*w') = l(w^*) + l(w')$.*

We denote the set of the elements w^* as in Lemma 4.2 by W^J, which is a set of representatives of W/W_J in W. There exist the following parabolic Bruhat/Birkhoff decompositions.

Proposition 4.3 ([12,15,22]). *Let J be a subset of I and, W_J and W^J be as above. Then we have*

$$G = \bigcup_{w^*\in W^J} U^{+}\overline{w^*}P_J, \qquad G = \bigcup_{w^*\in W^J} U^{-}\overline{w^*}P_J.$$

4.2. Unipotent crystal structure on Schubert variety

For $\Lambda \in P_+$ (P_+ is the set of dominant integral weight), let us denote an integral highest weight simple module with the highest weight Λ by $L(\Lambda)$ [6] and its projective space by $\mathbb{P}(\Lambda) := (L(\Lambda) \setminus \{0\})/\mathbb{C}^{\times}$. Let $v_\Lambda \in \mathbb{P}(\Lambda)$ be the point corresponding to the line containing the highest weight vector of $L(\Lambda)$ and set

$$X(\Lambda) := G \cdot v_\Lambda \subset \mathbb{P}(\Lambda).$$

Set $J_\Lambda := \{i \in I | \langle h_i, \Lambda\rangle = 0\}$. By Proposition 4.3 and the fact that P_{J_Λ} is the stabilizer of v_Λ, we have the isomorphism between $X(\Lambda)$ and the flag variety G/P_{J_Λ}.

Proposition 4.4 ([15,22]). *There is the following isomorphism and the decomposition*;

$$\begin{aligned}\rho : G/P_{J_\Lambda} = \bigcup_{w\in W^{J_\Lambda}} U^{\pm}\overline{w}P_{J_\Lambda}/P_{J_\Lambda} &\xrightarrow{\sim} X(\Lambda)\\ g \cdot P_{J_\Lambda} &\mapsto g \cdot v_\Lambda\end{aligned}$$

Definition 4.5. We denote the image $\rho(U^+\overline{w}P_{J_\Lambda}/P_{J_\Lambda})$ (resp. $\rho(U^-\overline{w}P_{J_\Lambda}/P_{J_\Lambda})$) by $X(\Lambda)_w$ (resp. $X(\Lambda)^w$) and call it a *finite* (resp. co-finite) *Schubert cell* and its Zariski closure in $\mathbb{P}(\Lambda)$ by $\overline{X}(\Lambda)_w$ (resp. $\overline{X}(\Lambda)^w$) and call it a *finite* (resp. co-finite) *Schubert variety*.

The names "finite" and "co-finite" come from the fact

$$\dim X(\Lambda)_w = l(w), \qquad \mathrm{codim}_{X(\Lambda)} X(\Lambda)^w = l(w),$$

Indeed, $X(\Lambda)_w \cong \mathbb{C}^{l(w)}$. There exist the following closure relations:

$$\overline{X}(\Lambda)_w = \bigsqcup_{y\le w,\, y\in W^{J_\Lambda}} X(\Lambda)_y, \qquad \overline{X}(\Lambda)^w = \bigsqcup_{y\ge w,\, y\in W^{J_\Lambda}} X(\Lambda)^y. \tag{4.3}$$

Indeed, by [12, 7.1, 7.3]:

$$\overline{X}(\Lambda)_w \text{ and } \overline{X}(\Lambda)^w \text{are ind-varieties.} \tag{4.4}$$

Let us associate a unipotent crystal structure with $X(\Lambda)_w$. Since by the definition of $X(\Lambda)_w$ and Proposition 4.4, we have $X(\Lambda)_w = U^+\overline{w}\cdot v_\Lambda$, the following lemma.

Lemma 4.6. *Schubert cell $X(\Lambda)_w$ is a U-variety.*

Next, let us construct a U-morphism $X(\Lambda)_w \to B^-$. For that purpose, we consider the following: let $w = s_{i_1}s_{i_2}\cdots s_{i_k}$ be a reduced expression and set $U_w = U\cap \overline{w}U^-\overline{w}^{-1}$ and $U^w = U\cap\overline{w}U\overline{w}^{-1}$. Define

$$\beta_1 = \alpha_{i_1}, \qquad \beta_2 = s_{i_1}(\alpha_{i_2}), \ldots, \beta_k = s_{i_1}s_{i_2}\cdots s_{i_{k-1}}(\alpha_{i_k}),$$

then we have

$$U_w := U_{\beta_1}\cdot U_{\beta_2}\cdots U_{\beta_k}.$$

This is a closed subgroup of U and we have an isomorphism of ind (algebraic)-varieties [22]

$$U_w \cong U_{\beta_1}\times U_{\beta_2}\times\cdots\times U_{\beta_k} \cong \mathbb{C}^k, \tag{4.5}$$

by

$$\begin{aligned} &U_w\cdot\overline{w} = U_{\alpha_{i_1}}\overline{s}_{i_1}\cdot U_{\alpha_{i_2}}\overline{s}_{i_2}\cdots\cdot U_{\alpha_{i_k}}\overline{s}_{i_k} \xrightarrow{\sim} \mathbb{C}^k,\\ &\quad x_{i_1}(a_1)\overline{s}_{i_1}\cdot x_{i_2}(a_2)\overline{s}_{i_2}\cdots\cdot x_{i_k}(a_k)\overline{s}_{i_k} \mapsto (a_1,a_2,\ldots,a_k). \end{aligned} \tag{4.6}$$

Lemma 4.7 ([22, 2.2]).

(i) *We have a decomposition*

$$U = U_w\cdot U^w \tag{4.7}$$

and this decomposition is unique in the sense; if $u_1v_1 = u_2v_2(u_i \in U_w, v_i \in U^w)$, *then* $u_1 = u_2$ *and* $v_1 = v_2$.

(ii) *For any* $w \in W^{J_\Lambda}(\Lambda \in P_+)$, *there exists an isomorphism of ind (algebraic)-varieties*

$$\begin{aligned}\delta : U_w &\xrightarrow{\sim} X(\Lambda)_w\\ u &\mapsto u \cdot v_\Lambda\end{aligned}$$

The following lemma is the first step for our purpose.

Lemma 4.8. *For any* $u \in U$ *and* $w \in W$, *there exist unique* $u' \in U_w \cdot \overline{w}$ *and* $v \in U$ *such that* $u\overline{w} = u'v$.

Proof. By Lemma 4.7(i), there are unique $u'' \in U_w$ and $v'' \in U^w$ such that $u = u''v''$. By the definition $U^w = U \cap \overline{w}U\overline{w}^{-1}$, we have $\overline{w}^{-1}v''\overline{w} \in U$. Thus, setting $u' = u''\overline{w}$ and $v = \overline{w}^{-1}v''\overline{w}$, we get the desired result. □

By using this decomposition, we define the following rational morphisms;

$$\begin{aligned}p_w : U \cdot \overline{w} &\longrightarrow U_w \cdot \overline{w}\\ u\overline{w} &\mapsto u'\\ p^w : U \cdot \overline{w} &\longrightarrow U\\ u\overline{w} &\mapsto v\end{aligned}$$

Define a rational U-action on $U_w \cdot \overline{w}$ by

$$\begin{aligned}&U \times U_w \cdot \overline{w} \longrightarrow U_w \cdot \overline{w}\\ &(x, u\overline{w}) \mapsto x(u\overline{w}) := p_w(xu\overline{w}) = xu\overline{w} \cdot p^w(xu\overline{w})^{-1}\end{aligned}$$

Next, we show the following lemma.

Lemma 4.9. *Let* $\pi^- : G \to B^-$ *and* $\alpha_{B^-} : U \times B^- \to B^-$ *be as in Section 3.2. For* $x \in U$ *and* $u\overline{w} \in U_w\overline{w}$, *we have*

$$\alpha_{B^-}(x, \pi^-(u\overline{w})) = \pi^-(x(u\overline{w})).$$

Proof. We have

$$\begin{aligned}\pi^-(x(u\overline{w})) &= x(u\overline{w}) \cdot \pi(x(u\overline{w}))^{-1} = xu\overline{w} \cdot p^w(xu\overline{w})^{-1}\pi(xu\overline{w} \cdot p^w(xu\overline{w})^{-1})^{-1}\\ &= xu\overline{w} \cdot p^w(xu\overline{w})^{-1}p^w(xu\overline{w})\pi(xu\overline{w})^{-1} \quad (\text{since } p^w(xu\overline{w}) \in U)\\ &= xu\overline{w} \cdot \pi(xu\overline{w})^{-1} = \pi^-(xu\overline{w})\end{aligned}$$

On the other hand,

$$
\begin{aligned}
\alpha_{B^-}(x,\pi^-(u\overline{w})) &= \pi^-(x\pi^-(u\overline{w})) = x\pi^-(u\overline{w})\cdot\pi(x\pi^-(u\overline{w}))^{-1}\\
&= xu\overline{w}\cdot\pi(u\overline{w})^{-1}\cdot\pi(xu\overline{w}\cdot\pi(u\overline{w})^{-1})^{-1}\\
&= xu\overline{w}\cdot\pi(u\overline{w})^{-1}\cdot\pi(u\overline{w})\cdot\pi(xu\overline{w})^{-1} \qquad (\text{since } \pi(u\overline{w})\in U)\\
&= xu\overline{w}\cdot\pi(xu\overline{w})^{-1} = \pi^-(xu\overline{w}),
\end{aligned}
$$

which completes the proof. □

Define an isomorphism of ind (algebraic)-varieties

$$\zeta : X(\Lambda)_w \xrightarrow{\sim} U_w\overline{w}$$

$$v \mapsto \zeta(v) := \delta^{-1}(v)\overline{w},$$

where $w\in W^{J_\Lambda}$ and $\Lambda\in P_+$. Since $X(\Lambda)_w$ is U-orbit of $\rho(\overline{w}\cdot P_{J_\Lambda}/P_{J_\Lambda})$, U acts rationally on $X(\Lambda)_w$. We denote the action of $x\in U$ on $v\in X(\Lambda)_w$ by $x(v)$.

Lemma 4.10. *The isomorphism $\zeta : X(\Lambda)_w\to U_w\overline{w}$ is a U-morphism.*

Proof. It is sufficient to show that $\zeta(x(v)) = x(\zeta(v))$ for $x\in U$ and $v\in X(\Lambda)_w$. Set $u = \delta^{-1}(v)$ and then we have $v = u\overline{w}v_\Lambda$. Since v_Λ is stable by the action of U, i.e., $U\cdot v_\Lambda = v_\Lambda$, we get

$$x(v) = p_w(xu\overline{w})(v_\Lambda).$$

Since $p_w(xu\overline{w})\in U_w\overline{w}$, we get

$$\zeta(x(v)) = p_w(xu\overline{w}).$$

We also have $x(\zeta(v)) = x(u\overline{w}) = p_w(xu\overline{w})$ and then $\zeta(x(v)) = x(\zeta(v))$. □

Define a rational morphism $f_w : X(\Lambda)_w\to B^-$ by $f_w = \pi^-\circ\zeta$. The following is one of the main results of this article.

Theorem 4.11. *For $\Lambda\in P_+$ and $w\in W^{J_\Lambda}$,* let *$X(\Lambda)_w$ be a finite Schubert cell and $f_w : X(\Lambda)_w\to B^-$ be as defined above. Then the pair $(X(\Lambda)_w, f_w)$ is a unipotent G-crystal.*

Proof. We see that $X(\Lambda)_w$ is a U-variety in Lemma 4.6. So, we may show that f_w is a U-morphism. For $x\in U$ and $v\in X(\Lambda)_w$, we get

$$f_w(x(v)) = \pi^-(\zeta(x(v))) = \pi^-(x(\zeta(v))) = \pi^-(x(u\overline{w})),$$

where $u = \delta^{-1}(v)$. On the other hand,

$$x(f_w(v)) = x(\pi^-(\zeta(v))) = x(\pi^-(u\overline{w})) = \alpha_{B^-}(x,\pi^-(u\overline{w})).$$

By Lemma 4.9, we obtain $f_w(x(v)) = x(f_w(v))$, which implies that f_w is a U-morphism. □

In the sense of Definition 3.5(ii), ζ is an isomorphism of unipotent crystals on $X(\Lambda)_w$ and $U_w\overline{w}$.

Since $X(\Lambda)_w \hookrightarrow \overline{X}(\Lambda)_w$ is an open embedding, they are birationally equivalent. Let $\omega : \overline{X}(\Lambda)_w \to X(\Lambda)_w$ be the inverse birational isomorphism. Thus, $\overline{f}_w := f_w \circ \omega : \overline{X}(\Lambda)_w \to B^-$ is a U-morphism. Then we have

Corollary 4.12. *Let $\overline{X}(\Lambda)_w$ be a finite Schubert variety and $\overline{f}_w$ be defined as above. Then the pair $(\overline{X}(\Lambda)_w, \overline{f}_w)$ is a unipotent G-crystal.*

Remark. Note that for all $w \leq w'$, we have the closed embedding $\overline{X}(\Lambda)_w \hookrightarrow \overline{X}(\Lambda)_{w'}$ [22], and the isomorphism

$$X(\Lambda) \xrightarrow{\sim} \varinjlim_{w \in W^{J_\Lambda}} \overline{X}(\Lambda)_w.$$

Nevertheless, in general, we do not obtain a unipotent crystal structure on $X(\Lambda)$ by using this direct limit since for $y < w$, the rational morphism $\overline{f}_w : \overline{X}(\Lambda)_w \to B^-$ is not defined on $\overline{X}(\Lambda)_y$.

4.3. Geometric crystal structure on $X(\Lambda)_w$

As we have seen in Section 3.3, we can associate geometric crystal structure with the finite Schubert cell (resp. variety) $X(\Lambda)_w$ (resp. $\overline{X}(\Lambda)_w$) since we have shown that they are unipotent G-crystals.

Now, let us verify the condition by which the function $\varphi_i : X(\Lambda)_w \to \mathbb{C}$ is not identically zero.

We recall the formula:

$$x_i(a)y_j(b) = \begin{cases} y_i\left(\dfrac{b}{1+ab}\right)\alpha_i^\vee(1+ab)x_i\left(\dfrac{a}{1+ab}\right) & \text{if } i = j, \\ y_j(b)x_i(a) & \text{if } i \neq j. \end{cases} \tag{4.8}$$

Hence, we have

$$x_i(c)\overline{s}_i = y_i\left(\frac{1}{c}\right)\alpha_i^\vee(c)x_i\left(-\frac{1}{c}\right), \tag{4.9}$$

where $\overline{s}_i = x_i(-1)y_i(1)x_i(-1)$. We also have

$$\alpha_i^\vee(a)x_j(b) = x_j(a^{a_{ij}}b)\alpha_i^\vee(a), \qquad \alpha_i^\vee(a)y_j(b) = y_j(a^{-a_{ij}}b)\alpha_i^\vee(a). \tag{4.10}$$

By the formula (4.8), (4.9), and (4.10), we obtain

$$x_i(a)\cdot\left(y_j\left(\frac{1}{c}\right)\alpha_j^\vee(c)\right) = \left(y_j\left(\frac{1}{c}\right)\alpha_j^\vee(c)\right)\cdot x_i(c^{-a_{ji}}a) \quad (i \neq j), \tag{4.11}$$

$$x_i(a)\cdot\left(y_i\left(\frac{1}{c}\right)\alpha_i^\vee(c)\right) = \left(y_i\left(\frac{1}{a+c}\right)\alpha_i^\vee(a+c)\right)\cdot x_i\left(\frac{a}{ac+c^2}\right). \tag{4.12}$$

Due to these formula, we get the following lemma.

Lemma 4.13. *For $w = s_{i_1}s_{i_2}\cdots s_{i_k} \in W$ (reduced expression) and $c_1, c_2, \dots, c_k \in \mathbb{C}^\times$, there exist $c'_1, c'_2, \dots, c'_k$ such that*

$$\begin{aligned}&\pi^-(x_{i_1}(c_1)\overline{s}_{i_1} \cdot x_{i_2}(c_2)\overline{s}_{i_2} \cdots x_{i_k}(c_k)\overline{s}_{i_k})\\ &\quad = y_{i_1}\left(\frac{1}{c'_1}\right)\alpha^\vee_{i_1}(c'_1)\cdot y_{i_2}\left(\frac{1}{c'_2}\right)\alpha^\vee_{i_2}(c'_2)\cdots y_{i_k}\left(\frac{1}{c'_k}\right)\alpha^\vee_{i_k}(c'_k).\end{aligned} \tag{4.13}$$

For $w \in W$, let $w = s_{i_1}s_{i_2}\cdots s_{i_k}$ be one reduced expression and set

$$I(w) := \{i_1, i_2, \dots, i_k\}.$$

Indeed, this does not depend on the choice of reduced expressions since W is a Coxeter group. By Lemma 4.13, we get

Lemma 4.14. *For $w \in W$ and $i \in I$, if $i \in I(w)$, then the function $\varphi_i : X(\Lambda)_w \to \mathbb{C}$ is not identically zero.*

Now, by Theorem 3.8, we have

Theorem 4.15. *For $w \in W$, suppose that $I = I(w)$. We can associate the geometric G-crystal structure with the finite Schubert cell $X(\Lambda)_w$ (resp. variety $\overline{X}(\Lambda)_w$) by setting (see* (3.4) *and* (3.5))

$$\gamma_w := \mathrm{proj}_T \circ f_w(\text{resp. } \overline{\gamma}_w := \mathrm{proj}_T \circ \overline{f}_w), \qquad e_i^c(x) = x_i\left(\frac{c-1}{\varphi_i(x)}\right)(x),$$

where $\mathrm{proj}_T : B^- = U^-T \to T$.

We denote this induced geometric crystal by $(X(\Lambda)_w, \gamma_w, \{e_i\}_{i\in I})$ (resp. $(\overline{X}(\Lambda)_w, \overline{\gamma}_w, \{e_i\}_{i\in I})$). This geometric/unipotent crystal $(X(\Lambda)_w, \gamma_w, \{e_i\}_{i\in I})$ is realized in B^- in the following sense.

Proposition 4.16. *For $w = s_{i_1}\cdots s_{i_k}$, define*

$$B^-_w := \{Y_w(c_1, \dots, c_k) := y_{i_1}\left(\frac{1}{c_1}\right)\alpha^\vee_{i_1}(c_{i_1})\cdots y_{i_k}\left(\frac{1}{c_k}\right)\alpha^\vee_{i_k}(c_{i_k}) \in B^- | c_i \in \mathbb{C}^\times\}.$$

and U-actions on B^-_w by

$$u(Y_w(c_1, \dots, c_k)) := \pi^-(u \cdot Y_w(c_1, \dots, c_k)) \quad (u \in U).$$

Then $X(\Lambda)_w$ and B^-_w are birationally equivalent via f_w and isomorphic as unipotent crystals. Moreover, they are isomorphic as induced geometric crystals.

Proof. By Lemma 4.13, it is sufficient to show that they are birationally equivalent to each other and then we may show that $U_w \cdot w$ and B_w^- are birationally equivalent via π^-. For that purpose, since we have the isomorphism (4.6) and the birational isomorphism $B_w^- \cong (\mathbb{C}^\times)^k$, it suffices to show that the correspondence $(c_1, \ldots, c_k) \longleftrightarrow (c'_1, \ldots, c'_k)$ in (4.13) is birational. In (4.13), each c'_i is a rational function in $c_1, c_2, \ldots, c_i$ obtained by composing the birational morphisms defined by (4.11) and (4.12) (in particular, $c_1 = c'_1$), which implies that $U_w \cdot w$ and B_w^- are birationally equivalent. $\square$

Example 4.17. We consider the case $G = SL_{n+1}(\mathbb{C})$, i.e., the Cartan matrix $A = (a_{ij})_{i,j\in I}$ is given by; $a_{ii} = 2$, $a_{ii\pm1} = -1$ and $a_{ij} = 0$ otherwise. Here $I = \{1, 2, \ldots, n\}$. Take $\tilde{w} = s_1 s_2 \cdots s_n \in W$. In this case, we can easily find that $I = I(\tilde{w})$ and

$$\begin{aligned}&\pi^-(x_1(c_1)\overline{s}_1 x_2(c_2)\overline{s}_2 \cdots x_n(c_n)\overline{s}_n)\\ &\quad = y_1\left(\frac{1}{c_1}\right)\alpha_1^\vee(c_1) y_2\left(\frac{1}{c_2}\right)\alpha_2^\vee(c_2)\cdots y_n\left(\frac{1}{c_n}\right)\alpha_n^\vee(c_n).\end{aligned}$$

Here changing the coordinate by $c_i = a_1 a_2 \cdots a_i$ and identifying $y_i(a) = I_n + aE_{i+1\,i}$, we obtain

$$f_{\tilde{w}}(X(\Lambda)_{\tilde{w}}) = \left\{ u(a) := \begin{pmatrix} a_1 & & & & & \\ 1 & a_2 & & & & \\ & 1 & \cdot & & & \\ & & \cdot\cdot & & & \\ & & & \cdot\cdot & & \\ & & & & a_n & \\ & & & & 1 & \frac{1}{a_1\cdots a_n} \end{pmatrix} ; a_i \in \mathbb{C}^\times \right\}$$

where $a = (a_1, \ldots, a_{n+1})$ and $a_1 a_2 \cdots a_{n+1} = 1$. By using this explicit presentation, we describe the geometric crystal structure of $f_{\tilde{w}}(X(\Lambda)_{\tilde{w}})$. Since $\varphi_i(u(a)) = 1/a_i$, we have

$$\begin{aligned} e_i^c(u(a)) &= x_i(a_i(c-1)) \cdot u(a) \cdot x_i(a_{i+1}(c^{-1}-1)) \\ &= u(a_1, \ldots, ca_i, c^{-1}a_{i+1}, \ldots, a_{n+1}). \end{aligned}$$

Furthermore, we have

$$\gamma_{\tilde{w}}(x_1(c_1)\overline{s}_1 x_2(c_2)\overline{s}_2 \cdots x_n(c_n)\overline{s}_n) = \alpha_1^\vee(c_1)\alpha_2^\vee(c_2)\cdots\alpha_n^\vee(c_n).$$

5. Tropicalization of crystals and Schubert varieties

We use the same notations as in the previous sections unless otherwise stated. We introduce a positive structure on geometric crystals and their ultra-discretizations and tropicalizations following [1, Sect. 2.5].

Let T be an algebraic torus over $\mathbb{C}$ and $X^*(T)$ (resp. $X_*(T)$) be the lattice of characters (resp. co-characters) of T. Let $R := \mathbb{C}[[c]][c^{-1}]$ and set $L(T) := \{\phi \in \mathrm{Hom}(O_T, R)\}$ (O_T is the ring of regular functions on T), which is called a set of *formal loops* on T. Here we specify the discrete valuation

$$\begin{aligned} v : R \setminus \{0\} &\longrightarrow \mathbb{Z} \\ \textstyle\sum_{n>-\infty} a_n c^n &\mapsto -\min\{n \in \mathbb{Z} | a_n \neq 0\}. \end{aligned}$$

For any $\phi \in L(T)$, set $\deg_T(\phi) := v \circ \phi|_{X^*(T)}$. Since for $f_1, f_2 \in R \setminus \{0\}$

$$v(f_1 f_2) = v(f_1) + v(f_2), \tag{5.1}$$

$\deg_T(\phi)$ can be considered as an element in $X_*(T) = \mathrm{Hom}(X^*(T), \mathbb{Z})$. Hence, $\deg_T$ can be seen as a map $\deg_T : L(T) \to X_*(T)$. For any $\lambda^\vee \in X_*(T)$, define $L_{\lambda^\vee}(T) := \deg_T^{-1}(\lambda^\vee) \subset L(T)$. Since $\deg_T^{-1}(\lambda^\vee)$ has an irreducible pro-$\mathbb{C}$ variety structure and $L(T) = \bigsqcup_{\lambda^\vee \in X_*(T)} L_{\lambda^\vee}(T)$, the set of irreducible components $\pi_0(L(T)) = \{L_{\lambda^\vee}(T) | \lambda^\vee \in X_*(T)\}$ can be identified with $X_*(T)$, *i.e.*, $\deg_T$ induces the bijection $\tilde{\deg}_T : \pi_0(L(T)) \xrightarrow{1:1} X_*(T)$.

More explicitly, set $T = (\mathbb{C}^\times)^l$ and identify $L(T)$ with $(R^\times)^l$. For $\lambda^\vee(c) = (c^{m_1}, c^{m_2}, \ldots, c^{m_l})$ $(m_j \in \mathbb{Z})$, we have

$$L_{\lambda^\vee}(T) = \left\{ \left(b_1 c^{-m_1} + \sum_{n>-m_1} a_n c^n, \ldots, b_l c^{-m_l} + \sum_{n>-m_l} a_n c^n \right) : b_1, \ldots, b_l \neq 0 \right\}.$$

Let $f : T \to T'$ be a rational morphism between two algebraic tori T and T'. The morphism f induces the rational morphism $\tilde{f} : L(T) \to L(T')$ and then the map $\pi_0(\tilde{f}) : \pi_0(L(T)) \to \pi_0(L(T'))$, which defines the map $\deg(f) : X_*(T) \to X_*(T')$.

$$\begin{array}{ccc} \pi_0(L(T)) & \xrightarrow{\pi_0(\tilde{f})} & \pi_0(L(T')) \\ \downarrow{\scriptstyle \tilde{\deg}_T} & & \downarrow{\scriptstyle \tilde{\deg}_{T'}} \\ X_*(T) & \xrightarrow{\deg(f)} & X_*(T') \end{array}$$

A rational function $f(c) \in \mathbb{C}(c)$ $(f \neq 0)$ is *positive* if f can be expressed as a ratio of polynomials with positive coefficients.

Remark. A rational function $f(c) \in \mathbb{C}(c)$ is positive if and only if $f(a) > 0$ for any $a > 0$ (pointed out by M.Kashiwara).

If $f_1, f_2 \in \mathbb{C}(c)(\subset R)$ are positive, then we have

$$v(f_1 f_2) = v(f_1) + v(f_2), \tag{5.2}$$

$$v\left(\frac{f_1}{f_2}\right) = v(f_1) - v(f_2), \tag{5.3}$$

$$v(f_1 + f_2) = \max(v(f_1), v(f_2)). \tag{5.4}$$

Definition 5.1 ([1]). A rational morphism $f : T \to T'$ between two algebraic tori T, T' is called *positive*, if the following two conditions are satisfied:

(i) For any co-character $\lambda^\vee : \mathbb{C}^\times \to T$, the image of $\lambda^\vee$ is contained in dom(f).
(ii) For any co-character $\lambda^\vee : \mathbb{C}^\times \to T$ and any character $\mu : T' \to \mathbb{C}^\times$, the composition $\mu \circ f \circ \lambda^\vee$ is a positive rational function.

Denote by $\mathrm{Mor}^+(T, T')$ the set of positive rational morphisms from T to T'.

Lemma 5.2 ([1]). *For any positive rational morphisms* $f \in \mathrm{Mor}^+(T_1, T_2)$ *and* $g \in \mathrm{Mor}^+(T_2, T_3)$, *the composition* $g \circ f$ *is in* $\mathrm{Mor}^+(T_1, T_3)$.

By this lemma, we can define a category $\mathcal{T}_+$ whose objects are algebraic tori over $\mathbb{C}$ and arrows are positive rational morphisms.

Lemma 5.3 ([1]). *For any algebraic tori* T_1, T_2, T_3, *and positive rational morphisms* $f \in \mathrm{Mor}^+(T_1, T_2)$, $g \in \mathrm{Mor}^+(T_2, T_3)$, *we have*

$$\deg(g \circ f) = \deg(g) \circ \deg(f).$$

By this lemma, we obtain a functor

$$\begin{aligned}
&\mathcal{UD} : \mathcal{T}_+ \longrightarrow \mathfrak{Set}\\
&T \mapsto X_*(T)\\
&(f : T \to T') \mapsto (\deg(f) : X_*(T) \to X_*(T')))
\end{aligned}$$

Definition 5.4 ([1]). Let $\chi = (X, \gamma, \{e_i\}_{i\in I})$ be a geometric pre-crystal, T be an algebraic torus and $\theta : T \to X$ be a birational isomorphism. The isomorphism θ is called *positive structure* on χ if it satisfies

(i) the rational morphism $\gamma \circ \theta : T' \to T$ is positive.
(ii) For any $i \in I$, the rational morphism $e_{i,\theta} : \mathbb{C}^\times \times T' \to T'$ given by

$$e_{i,\theta}(c, t) := \theta^{-1} \circ e_i^c \circ \theta(t)$$

is positive.

Applying the functor $\mathcal{UD}$ to positive rational morphisms $e_{i,\theta} : \mathbb{C}^\times \times T' \to T'$ and $\gamma \circ \theta : T' \to T$ (the notations are as above), we obtain

$$\tilde{e}_i := \mathcal{UD}(e_{i,\theta}) : \mathbb{Z} \times X_*(T') \to X_*(T'), \qquad \tilde{\gamma} := \mathcal{UD}(\gamma \circ \theta) : X_*(T') \to X_*(T).$$

Now, for given positive structure $\theta : T' \to X$ on a geometric pre-crystal $\chi = (X, \gamma, \{e_i\}_{i\in I})$, we associate the triplet $(X_*(T'), \tilde{\gamma}, \{\tilde{e}_i\}_{i\in I})$ with a pre-crystal structure (see [1, 2.2]) and denote it by $\mathcal{UD}_{\theta,T'}(\chi)$. By Lemma 3.3, we have the following theorem.

Theorem 5.5. *For any geometric crystal* $\chi = (X, \gamma, \{e_i\}_{i\in I})$ *and positive structure* $\theta : T' \to X$, *the associated pre-crystal* $\mathcal{UD}_{\theta,T'}(\chi) = (X_*(T'), \tilde{\gamma}, \{\tilde{e}_i\}_{i\in I})$ *is a free* W*-crystal* (*see* [1, 2.2])

We call the functor $\mathcal{UD}$ "ultra-discretization" instead of "tropicalization" unlike in [1]. The term "tropicalization" here means the inverse tropicalization in [1]. More precisely, for a crystal B, if there exists a geometric crystal χ, an algebraic torus T in $\mathcal{T}_+$ and a positive structure θ on χ such that $\mathcal{UD}_{\theta,T}(\chi) \cong B$ as crystals, we call χ a *tropicalization* of B.

Now, we define certain positive structure on geometric crystal B_w^- ($I = I(w)$, and $w \in W^{J_\Lambda}$) and see that it turns out to be a tropicalization of (Langlands dual of) some Kashiwara's crystal.

For $i \in I$, let B_i be the crystal defined by (see, e.g. [7])

$$B_i := \{(x)_i | x \in \mathbb{Z}\},$$
$$\tilde{e}_i(x)_i = (x+1)_i,\ \tilde{f}_i(x)_i = (x-1)_i,\ \tilde{e}_j(x)_i = \tilde{f}_j(x)_i = 0\ (i \neq j),$$
$$wt(x)_i = x\alpha_i,\ \varepsilon_i(x)_i = -x,\ \varphi_i(x)_i = x,\ \varepsilon_j(x)_i = \varphi_j(x)_i = -\infty (i \neq j).$$

For $w = s_{i_1}s_{i_2}\cdots s_{i_k} \in W$ and $\mathbf{i} = (i_1, i_2, \ldots, i_k) \in R(w)$, we define the morphism $\theta_{\mathbf{i}} : (\mathbb{C}^\times)^k \to B_w^-$ by

$$\theta_{\mathbf{i}}(c_1, c_2, \ldots, c_k) := y_{i_1}\left(\frac{1}{c_1}\right)\alpha_{i_1}^\vee(c_1)\cdots y_{i_k}\left(\frac{1}{c_k}\right)\alpha_{i_k}^\vee(c_k). \tag{5.5}$$

Similar statements to the following proposition are given in [1, Theorem 2.11] for reductive cases. Here we show it for arbitrary Kac–Moody cases by direct methods.

Proposition 5.6.

(i) *For any* $\mathbf{i} \in R(w)(w \in W, I(w) = I)$, *the morphism* $\theta_{\mathbf{i}}$ *defined in* (5.5) *is a positive structure on the geometric crystal* B_w^-.
(ii) *Geometric crystal* B_w^- *is a tropicalization of the Langlands dual of the crystal* $B_{i_1} \otimes B_{i_2} \otimes \cdots \otimes B_{i_k}$ *with respect to the positive structure* $\theta_{\mathbf{i}}(c_1, c_2, \ldots, c_k)$, *or equivalently* $\mathcal{UD}(B_w^-) \cong$ *Langlands dual*$(B_{i_1} \otimes \cdots \otimes B_{i_k})$ *as crystals.*

Proof. It is clear that $\theta_{\mathbf{i}}$ is a birational isomorphism. Since the rational morphism $\gamma : B_w^- \to T$ is given by

$$\gamma\left(y_{i_1}\left(\frac{1}{c_1}\right)\alpha_{i_1}^\vee(c_1)\cdots y_{i_k}\left(\frac{1}{c_k}\right)\alpha_{i_k}^\vee(c_k)\right) = \alpha_{i_1}^\vee(c_1)\cdots\alpha_{i_k}^\vee(c_k),$$

we have that $\gamma \circ \theta_{\mathbf{i}}$ is positive. In order to show that $e_{i,\theta_{\mathbf{i}}} : \mathbb{C}^\times \times T' \to T'$ is positive, we see the explicit action of e_i^c on $Y_w(c_1, \ldots, c_k)$. First let us evaluate $\varphi_i(Y_w(c_1, \ldots, c_k))$.

Lemma 5.7. *For* $Y := y_{i_1}(a_1)\cdots y_{i_k}(a_k) \in U^-$, *we have*

$$\varphi_i(Y) = \sum_{i_j = i} a_{i_j}. \tag{5.6}$$

Proof. Let $\{j_1, j_2, \ldots, j_r\}(j_1 < j_2 < \cdots < j_r)$ be the set of indices such that $i_{j_m} = i$. Then we can write

$$Y = A_0 \cdot y_i(a_{i_{j_1}}) \cdot A_1 \cdot y_i(a_{i_{j_2}}) \cdot A_2 \cdot y_i(a_{i_{j_3}}) \cdots A_{r-1} \cdot y_i(a_{i_{j_r}}) \cdot A_r,$$

where $A_s := \prod_{j_s < p < j_{s+1}} y_{i_p}(a_{i_p})(j_0 = 0, j_{r+1} = k+1)$. Here we set

$$B_m := y_i\left(-\sum_{m<s\le r} a_{i_{j_s}}\right) \cdot A_m \cdot y_i\left(\sum_{m<s\le r} a_{i_{j_s}}\right),$$

Then we have

$$Y = y_i\left(\sum_{0<s\le r} a_{i_{j_s}}\right) \cdot (B_0 \cdot B_1 \cdots B_r). \tag{5.7}$$

Since $B_0 \cdot B_1 \cdots B_r$ is in $Y_{-\alpha_i}$ and the decomposition (5.7) is unique by Lemma 3.7, we have

$$\varphi_i(Y) = \sum_{0<s\le r} a_{i_{j_s}} = \sum_{i_j = i} a_{i_j},$$

which is the desired result. □

Set

$$C_j^* := (c_1^{a_{i_1,i_j}} c_2^{a_{i_2,i_j}} \cdots c_{j-1}^{a_{i_{j-1},i_j}} c_j)^{-1} \qquad \left(C_1^* = \frac{1}{c_1}\right),$$

where $a_{i,j}$ is an (i, j)-entry of the generalized Cartan matrix A. By (4.10), we have

$$Y_w(c_1, \ldots, c_k) = y_{i_1}(C_1^*) \cdots y_{i_k}(C_k^*)\alpha_{i_1}^\vee(c_1) \cdots \alpha_{i_k}^\vee(c_k).$$

Then by Lemma 5.7, we obtain

$$\varphi_i(Y_w(c_1, \ldots, c_k)) \sum_{i_j = i} C_{i_j}^* = \sum_{j=1}^{k} \frac{\delta_{i,i_j}}{c_1^{a_{i_1,i}} c_2^{a_{i_2,i}} \cdots c_{j-1}^{a_{i_{j-1},i}} c_j}. \tag{5.8}$$

For $c \in \mathbb{C}$ and $i \in I$, define $\{\overline{C}_j\}_{1\le j\le k}$ and $\{\tilde{C}_j\}_{0\le j\le k}$ recursively by

$$\overline{C}_0 = c, \qquad \tilde{C}_j = c_j + \delta_{i_j,i}\overline{C}_{j-1}, \qquad \overline{C}_j = \frac{\overline{C}_{j-1} \cdot c_j^{1-a_{i_j,i}}}{\tilde{C}_j}.$$

Then, by using (4.11) and (4.12) repeatedly, we obtain

$$x_i(c)(Y_w(c_1, \ldots, c_k)) = Y_w(\tilde{C}_1, \ldots, \tilde{C}_k). \tag{5.9}$$

It is easy to get the explicit form of $\overline{C}_j$:

$$\overline{C}_j = \frac{c\prod_{m=1}^{j} c_m^{1-a_{i_m,i}}}{\sum_{1\le m\le j, i_m=i} c\cdot D_m + \prod_{m=1}^{j} c_m},$$

where

$$D_m := c_1^{1-a_{i_1,i}}\cdots c_{m-1}^{1-a_{i_{m-1},i}}\cdot c_{m+1}\cdots c_{j-1}c_j.$$

Now, in (5.9) replacing c with $(c-1)/\varphi_i(Y_w(c_1,\ldots,c_k))$ and using (5.8), we obtain

$$e_i^c(Y_w(c_1,\ldots,c_k)) = x_i\left(\frac{c-1}{\varphi_i(Y_w(c_1,\ldots,c_k))}\right)(Y_w(c_1,\ldots,c_k))) =: Y_w(\mathcal{C}_1,\ldots,\mathcal{C}_k),$$

where

$$\mathcal{C}_j := c_j\cdot\frac{\sum_{1\le m\le j, i_m=i} c/c_1^{a_{i_1,i}}\cdots c_{m-1}^{a_{i_{m-1},i}}c_m + \sum_{j<m\le k, i_m=i} 1/c_1^{a_{i_1,i}}\cdots c_{m-1}^{a_{i_{m-1},i}}c_m}{\sum_{1\le m<j, i_m=i} c/c_1^{a_{i_1,i}}\cdots c_{m-1}^{a_{i_{m-1},i}}c_m + \sum_{j\le m\le k, i_m=i} 1/c_1^{a_{i_1,i}}\cdots c_{m-1}^{a_{i_{m-1},i}}c_m}. \tag{5.10}$$

By this formula, it is clear that $e_{i,\theta_{\mathbf{i}}}$ is positive. We have shown (i).

Next, in order to show (ii), we see the action of $\tilde{e}_i^c$ on $B_{i_1}\otimes\cdots\otimes B_{i_k}$. Take $b_{\mathbf{i}} = (b_1)_{i_1}\otimes\cdots\otimes(b_k)_{i_k}$ ($\mathbf{i} = (i_1,\ldots,i_k)$, $b_j\in\mathbb{Z}$). Since the action of $\tilde{e}_i$ on tensor products is described explicitly in [7], we obtain

$$\tilde{e}_i^c(b_{\mathbf{i}}) = (\beta_1)_{i_1}\otimes\cdots\otimes(\beta_k)_{i_k},$$

where $\beta_j - b_j$

$$\begin{aligned} &= \max\left(\max_{\substack{1\le m\le j,\\ i_m=i}}\left(c-b_m-\sum_{l<m} b_l a_{i,i_l}\right),\quad \max_{\substack{j<m\le k,\\ i_m=i}}\left(-b_m-\sum_{l<m} b_l a_{i,i_l}\right)\right)\\ &\quad -\max\left(\max_{\substack{1\le m<j,\\ i_m=i}}\left(c-b_m-\sum_{l<m} b_l a_{i,i_l}\right),\quad \max_{\substack{j\le m\le k,\\ i_m=i}}\left(-b_m-\sum_{l<m} b_l a_{i,i_l}\right)\right). \end{aligned} \tag{5.11}$$

Now, we know that (5.10) and (5.11) are related to each other by the tropicalization/ultra-discretization operations:

$$
\begin{array}{ccc}
\mathcal{C}_j & \underset{\text{tropicalization}}{\overset{\text{ultra-discretization}}{\rightleftarrows}} & \beta_j \\
c_j & \longleftrightarrow & b_j \\
x \cdot y & \longleftrightarrow & x + y \\
\frac{x}{y} & \longleftrightarrow & x - y \\
x + y & \longleftrightarrow & \max(x, y) \\
a_{i,j} & \underset{\text{Langlands dual}}{\longleftrightarrow} & a_{j,i}
\end{array}
$$

We have completed the proof of (ii). □

The formula similar to (5.10), (5.11) are given in [2, Sect.5.2.] for the longest element w_0 (in reductive cases).

The following formulae are an immediate consequence of Proposition 5.6 and Lemma 3.3, which are given implicitly in [7] and shown by direct method in [16].

Corollary 5.8. *On the crystal $B_{i_1} \otimes \cdots \otimes B_{i_k}$, we have for any $c_1, c_2 \in \mathbb{Z}_{\geq 0}$*

$$
\begin{aligned}
&\tilde{e}_i^{c_1}\tilde{e}_j^{c_2} = \tilde{e}_j^{c_2}\tilde{e}_i^{c_1} && if\ \langle \alpha_i^\vee, \alpha_j\rangle = 0,\\
&\tilde{e}_i^{c_1}\tilde{e}_j^{c_1+c_2}\tilde{e}_i^{c_2} = \tilde{e}_j^{c_2}\tilde{e}_i^{c_1+c_2}\tilde{e}_j^{c_1} && if\ \langle \alpha_i^\vee, \alpha_j\rangle = \langle \alpha_j^\vee, \alpha_i\rangle = -1,\\
&\tilde{e}_i^{c_1}\tilde{e}_j^{2c_1+c_2}\tilde{e}_i^{c_1+c_2}\tilde{e}_j^{c_2} = \tilde{e}_j^{c_2}\tilde{e}_i^{c_1+c_2}\tilde{e}_j^{2c_1+c_2}\tilde{e}_i^{c_1} && if\ \langle \alpha_i^\vee, \alpha_j\rangle = -1,\ \langle \alpha_j^\vee, \alpha_i\rangle = -2,\\
&\tilde{e}_i^{c_1}\tilde{e}_j^{2c_1+c_2}\tilde{e}_i^{3c_1+c_2}\tilde{e}_j^{3c_1+2c_2}\tilde{e}_i^{c_1+c_2}\tilde{e}_j^{c_2} = \tilde{e}_j^{c_2}\tilde{e}_i^{c_1+c_2}\tilde{e}_j^{3c_1+2c_2}\tilde{e}_i^{3c_1+c_2}\tilde{e}_j^{2c_1+c_2}e_i^{c_1} && if\ \langle \alpha_i^\vee, \alpha_j\rangle = -1,\ \langle \alpha_j^\vee, \alpha_i\rangle = -3.
\end{aligned}
$$

Remark. What we considered in Example 4.17 is a different kind of positive structure on $B^-_{\tilde{w}}$ where $\tilde{w} = s_1 s_2 \cdots s_n$. More precisely, we define a rational morphism:

$$
\begin{aligned}
&\tilde{\theta} : (\mathbb{C}^\times)^n \longrightarrow B^-_{\tilde{w}}\\
&(a_1, \ldots, a_n) \mapsto y_1(\tfrac{1}{c_1})\alpha_1^\vee(c_1)\cdots y_n(\tfrac{1}{c_n})\alpha_n^\vee(c_n),
\end{aligned}
$$

where $c_i = a_1 a_2 \cdots a_i$. Then it is easy to see that $\tilde{\theta}$ gives a positive structure on $B^-_{\tilde{w}}$. Indeed, since we have

$$
e_i^c(Y_{\tilde{w}}(c_1, \ldots, c_n)) = Y_{\tilde{w}}(c_1, \ldots, c_{i-1,} cc_i, c_{i+1}, \ldots, c_n),
$$

we obtain

$$
e_{i,\tilde{\theta}}(c, (a_1, \ldots, a_n, a_{n+1})) = (a_1, \ldots, ca_i, c^{-1}a_{i+1}, \ldots, a_n, a_{n+1}),
$$

where $a_1 \cdots a_{n+1} = 1$. The ultra-discretization of the geometric crystal on $B^-_{\tilde{w}}$ with respect to $\tilde{\theta}$ is as follows. Set $\tilde{B} := \{(x_1, \ldots, x_{n+1}) \in \mathbb{Z}^{n+1} | x_1 + \cdots + x_{n+1} = 0\}$ and for $x := (x_1, \ldots, x_{n+1}) \in \tilde{B}$, set

$$
\tilde{e}_i^c(x) = (x_1, \ldots, x_i + c, x_{i+1} - c, \ldots, x_{n+1}) \quad (c \in \mathbb{Z}_{\geq 0}),
$$

and $\tilde{f}_i^c = \tilde{e}_i^{-c}$. Then $\mathcal{UD}_{\tilde{\theta},\mathbb{C}^n}(B^-_{\tilde{w}}, \gamma, \{e_i\})$ is the Langlands dual of the crystal $\tilde{B}$. The crystal $\tilde{B}$ holds the similar structure to some limit of "crystal base for the symmetric tensor module".

6. Tropical braid-type isomorphisms

As an application of the tropicalization/ultra-discretization given in the previous section, we shall give a new proof of the braid-type isomorphisms of crystals [19]. In order to do it, let us give the "tropical braid-type isomorphism" (similar formula is given in [2]).

To prove the tropical braid-type isomorphism, we need the following well-known facts (see e.g., [3]).

Lemma 6.1. *We have the following identities*:

(i) *Type* A_2: *set* $y_{\alpha_i+\alpha_j}(t) = s_j y_i(t) s_j^{-1}$, *we have*

$$y_i(a)y_j(b) = y_{\alpha_i+\alpha_j}(ab)y_j(b)y_i(a). \tag{6.1}$$

(ii) *Type* $B_2(\langle\alpha_i^\vee, \alpha_j\rangle = -2, \langle\alpha_j^\vee, \alpha_i\rangle = -1)$: set $y_{\alpha_i+\alpha_j}(t) = s_j y_i(t) s_j^{-1}$ and $y_{2\alpha_i+\alpha_j}(t) = s_i y_j(t) s_i^{-1}$, we have

$$y_i(a)y_j(b) = y_{2\alpha_i+\alpha_j}(a^2b)y_{\alpha_i+\alpha_j}(ab)y_j(b)y_i(a), \tag{6.2}$$

$$y_i(a)y_{\alpha_i+\alpha_j}(b) = y_{2\alpha_i+\alpha_j}(2ab)y_{\alpha_i+\alpha_j}(b)y_i(a). \tag{6.3}$$

(iii) *Type* G_2 $(\langle\alpha_i^\vee, \alpha_j\rangle = -3, \langle\alpha_j^\vee, \alpha_i\rangle = -1)$: set $y_{\alpha_i+\alpha_j}(t) = s_j y_i(t) s_j^{-1}$, $y_{2\alpha_i+\alpha_j}(t) = s_i y_{\alpha_i+\alpha_j}(t) s_i^{-1}$, $y_{3\alpha_i+\alpha_j}(t) = s_i y_{2\alpha_i+\alpha_j}(-t) s_i^{-1}$ and $y_{3\alpha_i+2\alpha_j}(t) = s_j y_{3\alpha_i+\alpha_j}(t) s_j^{-1}$, we have

$$y_i(a)y_j(b) = y_{3\alpha_i+2\alpha_j}(a^3b^2)y_{3\alpha_i+\alpha_j}(a^3b)y_{2\alpha_i+\alpha_j}(a^2b)y_{\alpha_i+\alpha_j}(ab)y_j(b)y_i(a), \tag{6.4}$$

$$y_{\alpha_i+\alpha_j}(a)y_{2\alpha_i+\alpha_j}(b) = y_{3\alpha_i+2\alpha_j}(3ab)y_{2\alpha_i+\alpha_j}(b)y_{\alpha_i+\alpha_j}(a), \tag{6.5}$$

$$y_j(a)y_{3\alpha_i+\alpha_j}(b) = y_{3\alpha_i+2\alpha_j}(-ab)y_{3\alpha_i+\alpha_j}(b)y_j(a). \tag{6.6}$$

Proposition 6.2. (Tropical braid-type isomorphism) *We have the following identities*:

(i) *Type* A_2:

$$\begin{aligned} y_i\left(\frac{1}{c_1}\right)\alpha_i^\vee(c_1)y_j\left(\frac{1}{c_2}\right)\alpha_j^\vee(c_2)y_i\left(\frac{1}{c_3}\right)\alpha_i^\vee(c_3) = y_j\left(\frac{c_1}{c_1c_3+c_2}\right) \\ \alpha_j^\vee\left(\frac{c_1c_3+c_2}{c_1}\right)y_i\left(\frac{1}{c_1c_3}\right)\alpha_i^\vee(c_1c_3)y_j\left(\frac{c_1c_3+c_2}{c_1c_2}\right)\alpha_j^\vee\left(\frac{c_1c_2}{c_1c_3+c_2}\right). \end{aligned} \tag{6.7}$$

(ii) *Type* $B_2(\langle\alpha_i^\vee, \alpha_j\rangle = -2, \langle\alpha_j^\vee, \alpha_i\rangle = -1)$:

$$y_i\left(\frac{1}{c_1}\right)\alpha_i^\vee(c_1)y_j\left(\frac{1}{c_2}\right)\alpha_j^\vee(c_2)y_i\left(\frac{1}{c_3}\right)\alpha_i^\vee(c_3)y_j\left(\frac{1}{c_4}\right)\alpha_j^\vee(c_4)$$
$$= y_j\left(\frac{1}{d_1}\right)\alpha_j^\vee(d_1)y_i\left(\frac{1}{d_2}\right)\alpha_i^\vee(d_2)y_j\left(\frac{1}{d_3}\right)\alpha_j^\vee(d_3)y_i\left(\frac{1}{d_4}\right)\alpha_i^\vee(d_4),$$

where

$$d_1 = c_4 + \frac{1}{c_2}\left(c_3 + \frac{c_2}{c_1}\right)^2, \qquad d_2 = c_1c_4 + c_3 + \frac{c_1c_3^2}{c_2}, \tag{6.8}$$

$$\frac{1}{d_3} = \frac{1}{c_2} + \frac{1}{c_2^2c_4}\left(c_3 + \frac{c_2}{c_1}\right)^2, \qquad \frac{1}{d_4} = \frac{c_4}{c_3} + \frac{c_3}{c_2} + \frac{1}{c_1}. \tag{6.9}$$

(iii) *Type* G_2 ($\langle\alpha_i^\vee, \alpha_j\rangle = -3$, $\langle\alpha_j^\vee, \alpha_i\rangle = -1$):

$$y_i\left(\frac{1}{c_1}\right)\alpha_i^\vee(c_1)y_j\left(\frac{1}{c_2}\right)\alpha_j^\vee(c_2)y_i\left(\frac{1}{c_3}\right)\alpha_i^\vee(c_3)y_j\left(\frac{1}{c_4}\right)\alpha_j^\vee(c_4)y_i\left(\frac{1}{c_5}\right)$$
$$\alpha_i^\vee(c_5)y_j\left(\frac{1}{c_6}\right)\alpha_j^\vee(c_6) = y_j\left(\frac{1}{d_1}\right)\alpha_j^\vee(d_1)y_i\left(\frac{1}{d_2}\right)\alpha_i^\vee(d_2)y_j\left(\frac{1}{d_3}\right)$$
$$\alpha_j^\vee(d_3)y_i\left(\frac{1}{d_4}\right)\alpha_i^\vee(d_4)y_j\left(\frac{1}{d_5}\right)\alpha_j^\vee(d_5)y_i\left(\frac{1}{d_6}\right)\alpha_i^\vee(d_6), \tag{6.10}$$

where

$$d_1 = \frac{1}{c_2^2}\left(c_3 + \frac{c_2}{c_1}\right)^3 + \frac{1}{c_4}\left(c_5 + \frac{c_4}{c_3}\right)^3 + \frac{2c_4}{c_2} + \frac{3c_4}{c_1c_3} + \frac{3c_5}{c_1} + \frac{3c_3c_5}{c_2} + c_6, \tag{6.11}$$

$$d_2 = \frac{c_1}{c_4}\left(c_5 + \frac{c_4}{c_3}\right)^3 + \frac{c_1c_3}{c_2^3}\left(c_3 + \frac{c_2}{c_1}\right)^3 + \frac{3c_1c_3c_5}{c_2} + \frac{2c_1c_4}{c_2} + \frac{2c_4}{c_3}$$
$$+ c_1c_6 + 2c_5, \tag{6.12}$$

$$\frac{1}{d_5} = \frac{1}{c_6}\left(\frac{1}{c_4}\left(c_5 + \frac{c_4}{c_3}\right)^2 + \frac{c_3}{c_2} + \frac{1}{c_1}\right)^3$$
$$+ \frac{c_6}{c_4} + \frac{3c_3c_5}{c_2c_4} + \frac{3c_5}{c_1c_4} + \frac{3}{c_1c_3} + \frac{2}{c_2}, \tag{6.13}$$

$$\frac{1}{d_6} = \frac{1}{c_1} + \frac{c_3}{c_2} + \frac{1}{c_4}\left(c_5 + \frac{c_4}{c_3}\right)^2 + \frac{c_6}{c_5}, \qquad d_3 = \frac{c_2c_4c_6}{d_1d_5},$$
$$d_4 = \frac{c_1c_3c_5}{d_2d_6}. \tag{6.14}$$

Proof. By Lemma 6.1, immediately we obtain the A_2 and B_2 cases. The G_2 case is quite complicated to obtain the explicit form of d_j's. Using (4.10), (6.4), (6.5) and (6.6), we can write the both sides of (6.10) in the form:

$$y_{3\alpha_i+2\alpha_j}(A)y_{3\alpha_i+\alpha_j}(B)y_{2\alpha_i+\alpha_j}(C)y_{\alpha_i+\alpha_j}(D)y_j(E)y_i(F)\alpha_i^\vee(G)\alpha_j^\vee(H).$$

Then comparing the both sides, we get (6.11), (6.12), (6.13) and (6.14). □

By Proposition 6.2, we easily see that each d_j is a positive rational function in c_j's. Thus, the map

$$(c_1, c_2, \cdots) \mapsto y_j\left(\frac{1}{d_1}\right)\alpha_j^\vee(d_1)y_i\left(\frac{1}{d_2}\right)\alpha_i^\vee(d_2)\cdots$$

gives rise to positive structures on $B^-_{w_0}$ where w_0 is the longest element of the Weyl group of type A_2, B_2 or G_2. Then, if we consider the ultra-discretization of this positive structures, we obtain the so-called "braid-type isomorphisms" between the tensor products of the crystal B_i's([19]).

Proposition 6.3.

(i) *If* $\langle\alpha_i^\vee, \alpha_j\rangle = \langle\alpha_j^\vee, \alpha_i\rangle = 0$,

$$\begin{aligned}&\phi_{ij}^{(0)} : B_i \otimes B_i \xrightarrow{\sim} B_j \otimes B_i\\ &(x)_i \otimes (y)_j \mapsto (y)_j \otimes (x)_i.\end{aligned}$$

(ii) *If* $\langle\alpha_i^\vee, \alpha_j\rangle = \langle\alpha_j^\vee, \alpha_i\rangle = -1$:

$$\begin{aligned}&\phi_{ij}^{(1)} : B_i \otimes B_j \otimes B_i \xrightarrow{\sim} B_j \otimes B_i \otimes B_j,\\ &(z_1)_i\otimes(z_2)_j \otimes (z_3)_i \mapsto (\max(z_3, z_2-z_1))_j\otimes(z_1+z_3)_i \otimes (-\max(-z_1, z_3-z_2))_j.\end{aligned} \tag{6.15}$$

(iii) If $\langle\alpha_i^\vee, \alpha_j\rangle = -1$, $\langle\alpha_j^\vee, \alpha_i\rangle = -2$,

$$\begin{aligned}&\phi_{ij}^{(2)} : B_i \otimes B_j \otimes B_i \otimes B_j \xrightarrow{\sim} B_j \otimes B_i \otimes B_j \otimes B_i,\\ &(z_1)_i \otimes (z_2)_j \otimes (z_3)_i \otimes (z_4)_j \mapsto (Z_1)_j \otimes (Z_2)_i \otimes (Z_3)_j \otimes (Z_4)_i,\\ &Z_1 = \max(z_4, z_2 - 2z_1, 2z_3 - z_2),\\ &Z_2 = \max(z_1 + z_4, z_3, z_1 - z_2 + 2z_3),\\ &Z_3 = -\max(-z_2, -z_4 - 2z_1, -2z_2 + 2z_3 - z_4),\\ &Z_4 = -\max(-z_3 + z_4, -z_1, z_3 - z_2).\end{aligned} \tag{6.16}$$

(iv) *If* $\langle\alpha_i^\vee, \alpha_j\rangle = -1$, $\langle\alpha_j^\vee, \alpha_i\rangle = -3$:

$$\begin{aligned}
&\phi_{ij}^{(3)} : B_i \otimes B_j \otimes B_i \otimes B_j \otimes B_i \otimes B_j \xrightarrow{\sim} B_j \otimes B_i \otimes B_j \otimes B_i \otimes B_j \otimes B_i,\\
&(z_1)_i \otimes (z_2)_j \otimes (z_3)_i \otimes (z_4)_j \otimes (z_5)_i \otimes (z_6)_j\\
&\mapsto (Z_1)_j \otimes (Z_2)_i \otimes (Z_3)_j \otimes (Z_4)_i \otimes (Z_5)_j \otimes (Z_6)_i,\\
&Z_1 = \max(z_6, 3z_5 - z_4, -3z_3 + 2z_4, -2z_2 + 3z_3, -3z_1 + z_2),\\
&Z_2 = \max(z_1 + z_6, z_1 - z_4 + 3z_5, z_1 - 3z_3 + 2z_4, z_1 - 2z_2 + 3z_3,\\
&\quad -z_1 + z_3),\\
&Z_3 = z_2 + z_4 + z_6 - Z_1 - Z_5,\\
&Z_4 = z_1 + z_3 + z_5 - Z_2 - Z_6,\\
&Z_5 = -\max(-z_4 + z_6, -3z_4 + 6z_5 - z_6, -6z_3 + 3z_4 - z_6,\\
&\qquad\qquad -3z_2 + 3z_3 - z_6, -3z_1 - z_6),\\
&Z_6 = -\max(-z_1, -z_2 + z_3, -z_4 + 2z_5, -2z_3 + z_4, -z_5 + z_6).
\end{aligned} \tag{6.17}$$

We call $\phi_{ij}^{(k)} (k = 0, 1, 2, 3)$ a *braid-type isomorphism*.

Proof. The formula in (6.15), (6.16) and (6.17) are obtained by rewriting the ones in [19, Proposition 4.1]by using:

$$\begin{aligned}
&a_1 + (a_2 + (a_3 + (\cdots + (a_k)_+ \cdots)_+)_+)_+\\
&\quad = \max(a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots, a_1 + \cdots + a_k).
\end{aligned}$$

In (6.7), the ultra-discretizations of $(c_1c_3 + c_2)/c_1$, c_1c_3 and $c_1c_2/(c_1c_3 + c_2)$ are

$$\begin{aligned}
v\left(\frac{c_1c_3 + c_2}{c_1}\right) &= \max(v(c_1) + v(c_3), v(c_2)) - v(c_1) = \max(v(c_3), v(c_2) - v(c_1)),\\
v(c_1c_3) &= v(c_1) + v(c_3),\\
v\left(\frac{c_1c_2}{c_1c_3 + c_2}\right) &= v(c_1) + v(c_2) - \max(v(c_1) + v(c_3), v(c_2))\\
&= -\max(v(c_3) - v(c_2), -v(c_1))
\end{aligned}$$

Thus, replacing $v(c_i)$ with z_i, we obtain (6.15).

Similarly, considering the ultra-discretizations of d_i's in (6.8) and (6.9), we get (6.16). Here note that in Proposition 6.3(iii), we suppose $\langle \alpha_i^\vee, \alpha_j\rangle = -1$, $\langle \alpha_j^\vee, \alpha_i\rangle = -2$, which is the Langlands dual of the condition in Proposition 6.2(ii).

In order to get the formula (6.17), we consider, e.g., $v(d_1)$:

$$\begin{aligned}
v(d_1) = \max(&-2z_2 + 3z_3, -3z_1 + z_2, 3z_5 - z_4, -3z_3 + 2z_4, z_6,\\
&z_4 - z_2, z_4 - z_1 - z_3, z_5 - z_1, z_3 + z_5 - z_2) \quad (v(c_j) = z_j),
\end{aligned} \tag{6.18}$$

which seems to be different from Z_1 in (6.17). But, it is easy to see that both are same by the following simple formula:

For $m_1, \ldots, m_k \in \mathbb{R}$ and $t_1, \ldots, t_k \in \mathbb{R}_{\geq 0}$ satisfying $t_1 + \cdots + t_k = 1$, we have

$$\max\left(m_1, \ldots, m_k, \sum_{j=1}^{k} t_j m_j\right) = \max(m_1, \ldots, m_k)$$

Indeed, in (6.18) we have

$$z_4 - z_2 = \tfrac{1}{2}A_1 + \tfrac{1}{2}A_4, \qquad z_4 - z_1 - z_3 = \tfrac{1}{6}A_1 + \tfrac{1}{3}A_2 + \tfrac{1}{2}A_4,$$
$$z_5 - z_1 = \tfrac{1}{6}A_1 + \tfrac{1}{3}A_2 + \tfrac{1}{3}A_3 + \tfrac{1}{6}A_4, \qquad z_3 + z_5 - z_2 = \tfrac{1}{2}A_1 + \tfrac{1}{3}A_3 + \tfrac{1}{6}A_4,$$

where $A_1 := -2z_2 + 3z_3$, $A_2 := -3z_1 + z_2$, $A_3 := 3z_5 - z_4$ and $A_4 := -3z_3 + 2z_4$.

Hence we have $Z_1 = v(d_1)$. Others are obtained similarly. Thus, considering the Langlands dual, we get the desired result. $\square$

7. Affine perfect crystal B_∞ for $\widehat{\mathfrak{sl}}_2$

In this subsection, we see an application of ultra-discretization of geometric crystal on Schubert cells/varieties defined for $\widehat{SL_2}$. This application is valid for only affine case, since by ultra-discretization we obtain so-called (affinization of)"affine perfect crystals". In this sense, the result in this subsection has no counterpart corresponding to reductive cases.

Perfect crystals are defined for quantum affine algebras and they play an important role in studying solvable lattice models [9,10]. In [8], certain limit of perfect crystals are introduced, which is denoted B_∞. This has a remarkable properties: $B(\infty) \cong B(\infty) \otimes B_\infty$, where $B(\infty)$ is the crystal of the nilpotent subalgebra of quantum affine algebra $U_q^-(\mathfrak{g})$ (See also [20,21]).

Let us recall the affinization of B_∞ for $\widehat{\mathfrak{sl}}_2$. Set weight lattice $P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \mathbb{Z}\delta$, where Λ_i $(i = 0, 1)$ is a fundamental weight and δ is a basis vector of null roots. The simple roots are expressed by $\alpha_0 = \delta - 2(\Lambda_1 - \Lambda_0)$ and $\alpha_1 = 2(\Lambda_1 - \Lambda_0)$.

The affine crystal $\mathrm{Aff}(B_\infty)$ is defined as follows:

$$\mathrm{Aff}(B_\infty) := \{z^k(l) | k, l \in \mathbb{Z}\}, \qquad wt(z^k(l)) := k\delta + 2l(\Lambda_0 - \Lambda_1), \tag{7.1}$$

$$\varepsilon_0(z^k(l)) := -l, \qquad \varphi_0(z^k(l)) := l, \qquad \varepsilon_1(z^k(l)) := l, \qquad \varphi_1(z^k(l)) := -l, \tag{7.2}$$

$$\tilde{e}_0(z^k(l)) := z^{k+1}(l+1), \qquad \tilde{f}_0(z^k(l)) := z^{k-1}(l-1), \tag{7.3}$$

$$\tilde{e}_1(z^k(l)) := z^k(l-1), \qquad \tilde{f}_1(z^k(l)) := z^k(l+1). \tag{7.4}$$

Here note that $\tilde{f}_i = \tilde{e}_i^{-1}$.

Now, for $G = \widehat{SL_2}$, set

$$B^-_{s_0 s_1} = \{Y(c_0, c_1) := y_0\left(\frac{1}{c_0}\right)\alpha_0^\vee(c_0) y_1\left(\frac{1}{c_1}\right)\alpha_1^\vee(c_1) | c_0, c_1 \in \mathbb{C}^\times\},$$

as in Section 5, which is isomorphic to the Schubert cell $X(\Lambda)_{s_0s_1}$ as a geometric crystal. Now we consider the following positive structure on $B^-_{s_0s_1}$:

$$\begin{aligned}\theta_0 : (\mathbb{C}^\times)^2 &\longrightarrow B^-_{s_0s_1}\\ (k,l) &\mapsto Y\left(k, \tfrac{k}{l}\right)\end{aligned} \tag{7.5}$$

Proposition 7.1. *We have* $\mathcal{UD}_{\theta_0}(B^-_{s_0s_1}) \cong \mathrm{Aff}(B_\infty)$.

Note that the algebra $\widehat{\mathfrak{sl}}_2$ is self-Langlands dual.

Proof. First, let us see the actions of e_i^c on $Y(k, k/l)$ explicitly. By (5.8), we get

$$\varphi_0\left(Y\left(k, \frac{k}{l}\right)\right) = \frac{1}{k}, \qquad \varphi_1\left(Y\left(k, \frac{k}{l}\right)\right) = kl,$$

and then it follows from (4.11) and (4.12) that

$$e_0^c\left(Y\left(k, \frac{k}{l}\right)\right) = x_0(k(c-1))\left(Y\left(k, \frac{k}{l}\right)\right) = Y\left(ck, \frac{k}{l}\right), \tag{7.6}$$

$$e_1^c\left(Y\left(k, \frac{k}{l}\right)\right) = x_1\left(\frac{c-1}{kl}\right)\left(Y\left(k, \frac{k}{l}\right)\right) = Y\left(k, \frac{ck}{l}\right). \tag{7.7}$$

Therefore, $\gamma \circ \theta_0 : (\mathbb{C}^\times)^2 \to T$ and $e_{i,\theta_0} := \theta_0^{-1} \circ e_i^c \circ \theta_0 : (\mathbb{C}^\times)^2 \to (\mathbb{C}^\times)^2$ are described:

$$\gamma\left(Y\left(k, \frac{k}{l}\right)\right) = \alpha_0^\vee(k)\alpha_1^\vee\left(\frac{k}{l}\right),$$

$$e_{0,\theta_0} : (k,l) \mapsto (ck, cl), \qquad e_{1,\theta_0} : (k,l) \mapsto \left(k, \frac{l}{c}\right). \tag{7.8}$$

Thus, by applying $\mathcal{UD}$ we obtain

$$\begin{aligned} wt : \mathbb{Z}^2 &\longrightarrow X_*(T)\\ (k,l) &\mapsto k\alpha_0 + (k-l)\alpha_1 = k\delta + 2l(\Lambda_0 - \lambda_1),\end{aligned} \tag{7.9}$$

$$\begin{aligned}\tilde{e}_0^c : \mathbb{Z}^2 &\longrightarrow \mathbb{Z}^2,\\ (k,l) &\mapsto (k+c, l+c),\end{aligned} \tag{7.10}$$

$$\begin{aligned}\tilde{e}_1^c : \mathbb{Z}^2 &\longrightarrow \mathbb{Z}^2,\\ (k,l) &\mapsto (k, l-c),\end{aligned} \tag{7.11}$$

which coincide with (7.1) and the actions of $\tilde{e}_i^c$ in (7.3) and (7.4). □

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